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Dissection: A New Paradigm for Solving Bicomposite Search Problems

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Abstract

Combinatorial search problems are usually described by a collection of possible states, a list of possible actions which map each current state into some next state, and a pair of initial and final states. The algorithmic problem is to find a sequence of actions which maps the given initial state into the desired final state. In this paper, we introduce the new notion of *bicomposite search problems*, and show that they can be solved with improved combinations of time and space complexities by using a new algorithmic paradigm called *dissection*. To demonstrate the broad applicability of our new paradigm, we show how to use it in order to untangle Rubik's cube and to solve a typical NP-complete partition problem with algorithms which are better than any previously described algorithm for these problems.

1. INTRODUCTION

A central problem in the design of efficient algorithms is how to solve search problems, in which we are given a pair of states and a collection of possible actions, and we are asked to find how to get from the first state to the second state by performing some sequence of actions. In some cases, we only want to decide whether such a sequence exists at all, while in other cases it is clear that such sequences exist but we are asked to find the shortest possible sequence. Many search problems of this type have associated decision problems which are NP-complete, and thus we do not expect to find any polynomial time algorithms which can solve all their instances. However, what we hope to find are new exponential time algorithms whose exponents are smaller than in the best previously known algorithms. For example, the problem of breaking a cryptographic scheme whose key has $n = 100$ unknown bits cannot be solved in a practical amount of time via an exhaustive key search algorithm, since its time complexity of 2^n would be beyond reach even for the largest currently available data center. However, if we manage to find a better cryptanalytic attack whose running time is $2^{n/2}$, we can break the scheme with a modest effort in spite of the exponential nature of this complexity function. One trick which is often helpful in such situations is to find a tradeoff between the time and space complexities of the attack: Exhaustive search requires a lot of time but a negligible amount of memory, and thus a tradeoff which uses more memory (in the form of large tables of precomputed values) in order to reduce the time (by skipping many computational steps) will be very beneficial. For reasons which are explained in the extended version of this paper (available in Dinur et al.[2]), we usually consider the product of the amount of time and the amount of space required by the algorithm as the appropriate complexity measure that we try to minimize. In the example above, breaking the cryptosystem with $T = 2^n$ time and an $S = 1$ space is infeasible, breaking it with $T = 2^{2n/3}$ time and $S = 2^{n/3}$ space (whose product $TS = 2^n$ is the same as before) is better but still barely feasible, and breaking it in $T = 2^{n/2}$ time and $S = 2^{n/4}$ space (whose product $TS = 2^{3n/4}$ has a smaller exponent) is completely feasible.

A typical search problem is defined by a condition F (e.g., in the form of a CNF Boolean formula which is a conjunction of clauses) and a candidate solution X (e.g., in the form of a 0/1 assignment to all the variables in F), and the goal is to find among all the possible X at least one that satisfies the condition that $F(X)$ is true. Such a representation has no internal structure in the sense that it uses the full description of F and the full value of X in order to decide whether $F(X)$ is satisfied. However, we can usually replace the all-or-nothing choice of X by a sequence of smaller decisions. For example, we can start with the assignment of 0 to all the variables, and at each stage we can decide to flip the current value of one of the Boolean variables. At any intermediate point in this process F is in a state in which some of its clauses are satisfied and some are not, and our goal is to reach a state in which all the clauses are simultaneously satisfied. More generally, we say that a search problem is *composite* if solving it can be described by a sequence of atomic actions, which change the system from some initial state through a series of intermediate states until it reaches some final desired state. Most search problems have such a composite structure, which can be represented by the *execution matrix* described in Figure 1. The rows of this matrix represent states $S_0, S_1, \ldots$, and the solution X is represented by the sequence of actions $a_1, a_2, \ldots$ on the left side of the figure in which action a_i changes state S_{i-1} to state S_i. In many cases, the atomic actions are invertible operations over the states, which makes it possible to map S_{i-1} to S_i by using the action a_i, and to map S_i back to S_{i-1} by using the action a_i^{-1}. For example, in the case of Boolean formulas we are allowed to flip the current value of any one of the variables in X, but we can cancel its effect by flipping the same variable a second time,

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so in this case the action and its inverse happen to be the same. In this paper we consider only such invertible cases, which allow to split the search problem by applying some forward actions to the initial state, applying some inverse actions to the final state, and searching for a meet-in-the-middle (MITM) state which combines these parts. Our new dissection technique can be applied even when some of the actions are not invertible, but this makes its description more complicated and its complexity slightly higher, as discussed in Dinur et al.[2]

So far we have partitioned the execution matrix into multiple rows by dividing the solution process into a series of atomic actions. In order to apply our new dissection technique, we also have to partition the execution matrix into multiple columns by dividing each state S_i into smaller chunks $S_{i,j}$ which we call substates, as described in Figure 2. However, only partitions in which the substates can be manipulated independently of each other will be useful to us. We say that a search problem has a *bicomposite structure* if it can be described by an execution matrix in which the knowledge of the action a_i makes it possible to uniquely determine substate $S_{i,j}$ from $S_{i-1,j}$ and $S_{i-1,j}$ from $S_{i,j}$, even when we know nothing about the other substates in the matrix. This immediately implies that if we choose any rectangle of any dimensions within the execution matrix, knowledge of the substates $S_{i-1,j}, S_{i-1,j+1}, \ldots, S_{i-1,k}$ along its top edge and knowledge of the actions $a_i, a_{i+1}, \ldots, a_\ell$ to its left suffices in order to compute the substates $S_{\ell,j}, S_{\ell,j+1}, \ldots, S_{\ell,k}$ along its bottom edge, and vice versa.

Figure 1. An execution matrix of a composite search problem.

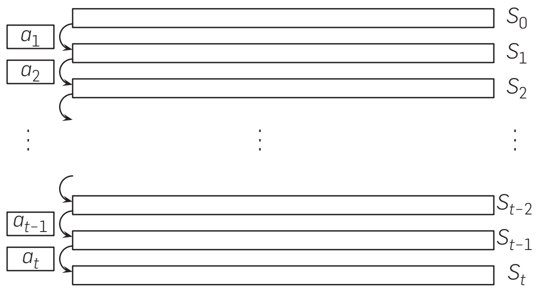
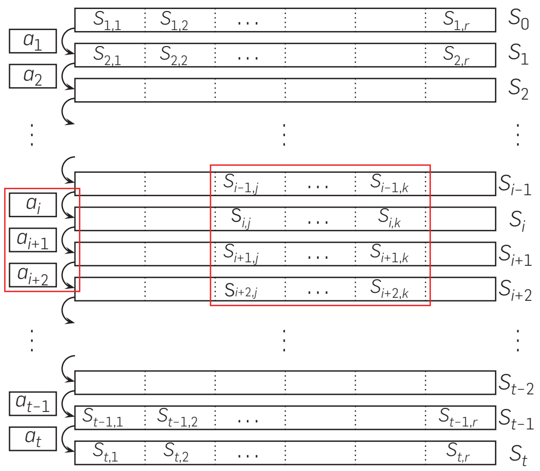


Figure 2. An execution matrix of a bicomposite search problem.


a_1
a_2
a_i
a_{i+1}
a_{i+2}
a_{t-1}
a_t
$S_{1,1}$ $S_{1,2}$ … $S_{1,r}$ S_0
$S_{2,1}$ $S_{2,2}$ … $S_{2,r}$ S_1
S_2
$S_{i-1,j}$ … $S_{i-1,k}$ S_{i-1}
$S_{i,j}$ … $S_{i,k}$ S_i
$S_{i+1,j}$ … $S_{i+1,k}$ S_{i+1}
$S_{i+2,j}$ … $S_{i+2,k}$ S_{i+2}
S_{t-2}
$S_{t-1,1}$ $S_{t-1,2}$ … $S_{t-1,r}$ S_{t-1}
$S_{t,1}$ $S_{t,2}$ … $S_{t,r}$ S_t

Not every search problem has such a bicomposite representation, but as we demonstrate in this paper there are many well-known problems which can be represented in such a way. When a problem is bicomposite, we can solve it with improved efficiency by using our generic new technique. Our main observation is that in such cases, we can improve the standard MITM algorithms (which try to match the forward and backward directions at a full intermediate state) by considering algorithms which only partially match the two directions at a partially specified intermediate state. In addition, we can reverse the logic of MITM algorithms, and instead of trying to converge from both ends of the execution toward some intermediate state which happens to be the same, we can start by partially guessing this intermediate state in all possible ways, and for each guessed value we can break the problem into two independent subproblems by proceeding from this intermediate state toward the two ends (exploiting the fact that in bicomposite problems we can determine the effect of long sequences of actions on partially specified states). We can then solve each one of these subproblems recursively by partially guessing additional substates in the execution matrix. We call this approach a *dissection* algorithm since it resembles the process in which a surgeon makes a sequence of short cuts at various strategically chosen locations in the patient's body in order to carry out the operation. For example, in Section 4 we show how to solve the well-known combinatorial partition problem by first guessing only two-sevenths of the bits of the state which occur after performing three-sevenths of the actions, and then solving one of the resultant subproblems by guessing in addition one-seventh of the bits of the state which occurs after performing five-sevenths of the actions.

Our main purpose in this paper is to introduce the new ideas by applying them in the simplest possible way to several well-known search problems. We intentionally overlook several nuisance issues whose proper handling is not easy to explain, and ignore several possible optimizations which can further reduce the complexity of our algorithms. When we analyze the running time of our algorithms, we often assume for the sake of clarity that the instance we try to solve is randomly chosen, and that the intermediate states we try to guess are uniformly distributed.

The paper is organized as follows. In Section 2 we describe the problem of solving Rubik's cube with the smallest possible number of steps as a bicomposite search problem, and in Section 3 we show how to solve it with time complexity which is approximately the square root of the size of the search space, and a space complexity which is approximately the fourth root of the size of the search space by using the simplest version of the new dissection algorithm. In Section 4 we describe several improvements of the basic dissection technique, and show how to use them in order to solve a different search problem called the "combinatorial partition problem" with combinations of time and space complexities which could not be achieved by any previously published algorithm, and which are more suitable for large scale FPGA-based hardware. We conclude the paper with some remarks in Section 5.

2. REPRESENTING RUBIK'S CUBE AS A BICOMPOSITE SEARCH PROBLEM

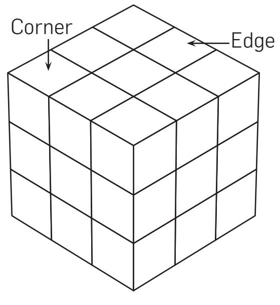
In this section we show how to construct a bicomposite representation of the well-known problem of solving a standard $3 \times 3 \times 3$ Rubik's cube.[6] We can assume that we always hold the cube in a fixed orientation, in which the white center color is at the top, the yellow center color is on the left, etc. One of the 27 subcubes is at the center of the cube, and we can ignore it since it is completely invisible. The six subcubes at the center of each face are not moved when we rotate that (or any other) face, and thus we can ignore them as well in our state representation. The actions we can take are to rotate each one of the six faces of the cube by 90, 180, or 270 degrees (we are not allowed to rotate a center slice since this will change the standard orientation of the cube defined above). Consequently, we have a repertoire of 18 atomic actions we can apply to each state of the cube. Note that all these actions are invertible mappings on the state of Rubik's cube in the sense that the inverse of a 90 degree rotation is a 270 degree rotation applied to the same face, and both of them are available as atomic actions.

Among the $27 - 6 - 1 = 20$ subcubes which we can move, 12 have two visible colors and are called edge subcubes, and 8 have three visible colors and are called corner subcubes (Figure 3). Each such subcube can be uniquely described by the combination of colors on it, such as a blue-white (BW) edge subcube or a green-orange-red (GOR) corner subcube. In addition, each location on the cube can be described by its relevant sides (i.e., a combination of top/bottom, left/right, front/back). We can thus describe any state of the cube by a vector of length 20, whose ith entry describes the current location of the ith subcube (e.g., the first entry in the vector will always refer to the blue-green edge subcube, and specify that it is currently located at the top-front position). To complete the specification, we also have to choose some standard orientation of the colors, and note that edge subcubes can be either in the standard (e.g., BW) or in an inverted (e.g., WB) state, and each corner subcube can be in one of three possible orientations (e.g., GOR, ORG, or RGO). Note that any possible action can only move edge subcubes to edge subcubes and corner subcubes to corner subcubes. If we use the first 12 positions in the state vector to describe the current locations of the 12 edge subcubes (in some fixed order), then each entry in these positions can be described by a number between 1 and 24 (specifying in which one of the 12 possible positions it is currently located and in which one of its 2 possible orientations). Similarly, when we use the last 8 positions in the state vector to describe the current locations of the 8 corner subcubes (in some fixed order), then each one of these entries can again contain a number between 1 and 24, this time specifying in which one of the 8 possible positions it is located and in which of its 3 possible orientations. We can thus describe any state of Rubik's cube by a vector of length 20 whose entries are numbers between 1 and 24. Any one of the 18 atomic actions will change 8 of the entries in this vector, by moving 4 edge subcubes and 4 corner subcubes to new positions and orientations, leaving the remaining 12 entries unchanged.

The problem of solving Rubik's cube can now be formalized as the following search problem: We are given a first vector of length 20 (representing the initial scrambled state of the cube) and a second vector of length 20 (representing the standard unscrambled state of the cube). We would like to find a sequence of atomic actions (in the form of face rotations) that will change the first vector into the second vector. Finding *some* sequence is easy, and there are many algorithms which are described in the recreational mathematics literature to achieve this (some of which are described in Slocum[6]). However, these algorithms are typically quite wasteful, using between 50 and 100 actions in order to move subcubes to their correct positions one at a time in some fixed order, ignoring the effect of these actions on other subcubes. Some algorithms use fewer actions, but then they are much harder to describe since they require a very detailed case analysis of the full state before choosing the first action. Recently, Davidson et al.[1] proved that 20 actions are necessary and sufficient if we want to solve *any* solvable state of Rubik's cube, but this was an existential result and the authors did not present an efficient way to actually find such a sequence. Note that the number of possible sequences of 20 atomic actions is $18^{20} = 12748236216396078174437376 \approx 2^{83}$, but we can slightly reduce the size of the search space to $18 \times 15^{19} = 399030807609558105468750 \approx 2^{78}$ by noticing that there is no point in rotating the same face twice in a row. However, even this reduced size cannot be exhaustively searched in a feasible amount of time.

To show that our representation of the search problem is bicomposite, assume that we know the current location and orientation of a particular subcube (namely, we know the value of $S_{i-1,j}$ in the execution matrix as a number between 1 and 24), and we apply to it some known face rotation action. We can then uniquely determine the new location and orientation of that particular subcube (namely, the value of $S_{i,j}$) even when we know nothing about the current location and orientation of any other subcube (namely, all the other $S_{k,\ell}$ values in the execution matrix). Notice that many other natural representations of the states of Rubik's cube do not have such a bicomposite structure. For example, if we associate the first entry in the state vector with a particular cube position (such as top-front) and use it to denote which edge subcube (such as BW) is currently

Figure 3. Rubik's cube.



located in it and in which orientation, then knowledge of just this entry in the first state does not tell us anything about which edge subcube (such as GR) replaces it at the top-front position if we rotate the top face by 90 degrees. Such a representation requires knowledge of other columns in the execution matrix, depending on which action was applied to the state, and thus we cannot use it in our new dissection technique.

As shown in Section 3, we can use the bicomposite representation in order to find for any given initial state of Rubik's cube a sequence of up to 20 face rotations using a completely feasible combination of a time complexity which is the square root of the size of the search space (namely, in about 2^{39} steps, or a few minutes on a standard PC) and a space complexity which is about the fourth root of this number (namely, about $2^{19.5}$ memory locations, or a few megabytes). The resultant algorithm is completely generic, makes no use of the details of the problem besides its bicompositeness, and matches the complexities of the best previous algorithm for solving Rubik's cube (designed about 25 years ago, see Fiat et al.[3]) which was highly specific and depended on the group-theoretic properties of the set of permutations defined by Rubik's cube.

3. THE BASIC DISSECTION TECHNIQUES

We now assume that we are given an initial state vector S_0 and a final state vector S_ℓ of Rubik's cube, and our goal is to find a series of atomic actions $a_1, a_2, \ldots, a_\ell$ that transform the initial state into the final state. As described in the previous section, we know that $\ell = 20$ suffices to find a solution, and hence our goal is to find $a_1, a_2, \ldots, a_{20}$.

Our dissection algorithms are extensions of the classical MITM algorithm, which was first presented in 1974 by Horowitz and Sahni[5] in order to solve the Knapsack problem. We can apply a MITM algorithm to almost any composite problem with invertible actions. When the size of the search space is 2^n, the MITM algorithm requires about $2^{n/2}$ time and $2^{n/2}$ space. For the sake of completeness, we describe below how to apply this algorithm in the context of Rubik's cube, whose search space has about 2^{78} states. In this case, a time complexity of $2^{78/2} = 2^{39}$ is feasible, but a space complexity of $2^{78/2} = 2^{39}$ random access memory locations is too expensive.

The complete details of the algorithm are given in Figure 4. The first step of the algorithm is to iterate over all possible 20/2 = 10 action sequences $a_1, \ldots, a_{10}$. We have $18 \times 15^9 \approx 2^{39}$ such sequences, and for each one, we apply its actions to S_0 and obtain a possible value for S_{10}. We store the 2^{39} values of $a_1, \ldots, a_{10}$ in a list next to the corresponding value of S_{10}, and sort[a] the list according to S_{10}. Next, we iterate over all possible action vectors $a_{11}, \ldots, a_{20}$. Again, there are about 2^{39} such action vectors, and for each one, we apply its inverted actions to S_{20} and obtain a possible value for S_{10}. We now search the sorted list for this value of S_{10}, and for each match, we obtain the corresponding value of $a_1, \ldots, a_{10}$ from the list and output $a_1, a_2, \ldots, a_{20}$ as a solution.

[a] For the sake of simplicity, we ignore logarithmic factors in our complexity analysis, and thus we assume that sorting is a linear time operation.

Figure 4. Meet-in-the-middle procedure to solve the Rubik's cube.

Algorithm MITM-Rubik

```
Input: Initial state $S_0$ and final state $S_{20}$
for all $a_1, a_2, \ldots, a_{10}$ do
    Compute $S_{10} = a_{10}(\ldots(a_2(a_1(S_0)))\ldots))$
    Store $(S_{10}, a_1, a_2, \ldots, a_{10})$ in a list $L$
Sort $L$ according to the value of $S_{10}$ in each entry (under some lexicographical order)
for all $a_{11}, a_{12}, \ldots, a_{20}$ do
    Compute $S_{10} = a_{11}^{-1}(\ldots(a_{19}^{-1}(a_{20}^{-1}(S_{20})))\ldots)$
    Search for $S_{10}$ in $L$
    if $S_{10}$ is found then
        return the associated $a_1, a_2, \ldots, a_{10}$ and $a_{11}, a_{12}, \ldots, a_{20}$ as a solution
```

The MITM algorithm requires about 2^{39} memory cells in order to store the sorted list, and its time complexity is about 2^{39}, which is the time required in order to iterate over each one of the action vectors $a_1, \ldots, a_{10}$ and $a_{11}, \ldots, a_{20}$.

3.1. Improving the MITM algorithm using dissection

In this section, we show how to improve the classical MITM algorithm on Rubik's cube by using a basic version of our new dissection technique. The main idea here is to "dissect" the execution matrix in the middle by iterating over all the possible values of some part of the middle state S_{10}. The size of the partial S_{10} that we iterate on is chosen such that it contains about $2^{n/4} = 2^{19.5}$ partial states. Since S_{10} is represented as a 20-entry vector, where each entry can attain 24 values, we choose to iterate on its first 4 entries, which can assume about $24^4 \approx 2^{18.5}$ values. For each such partial value of S_{10}, we use the bicomposite structure of the problem in order to independently work on the two partial execution matrices shown in Figure 5 as red and blue rectangles, and finally join the partial solutions in order to obtain the full action vector.

The complete details of the algorithm are given in Figure 6. We have an outer loop which iterates over all the possible values of $S_{10,1}, S_{10,2}, S_{10,3}, S_{10,4}$. Assuming that this value is correctly guessed, we first concentrate on the upper part of the execution matrix and find all the partial action vectors $a_1, \ldots, a_{10}$ which transform $S_{0,1}, S_{0,2}, S_{0,3}, S_{0,4}$ into $S_{10,1}, S_{10,2}, S_{10,3}, S_{10,4}$. This is done using a simple MITM algorithm on this smaller execution matrix. For each solution $a_1, \ldots, a_{10}$ that we obtain using the MITM algorithm, we apply its actions to the full state S_0, obtain candidate values of the full S_{10} state, and store it next to $a_1, \ldots, a_{10}$ in a list. After the MITM algorithm finishes populating the list, we sort it (e.g., in some lexicographic order) according to the value of S_{10}.

We now focus on the bottom execution matrix and find all the partial action vectors $a_{11}, \ldots, a_{20}$ which transform $S_{10,1}, S_{10,2}, S_{10,3}, S_{10,4}$ into $S_{20,1}, S_{20,2}, S_{20,3}, S_{20,4}$. We use the same idea that we used for the upper part, that is, we execute a MITM algorithm on the bottom execution matrix. For each solution $a_{11}, \ldots, a_{20}$ that we obtain, we apply its inverse actions to S_{20} and obtain a value for S_{10}. Then, we check for matches

Figure 5. Dissection of the Rubik's cube execution matrix.

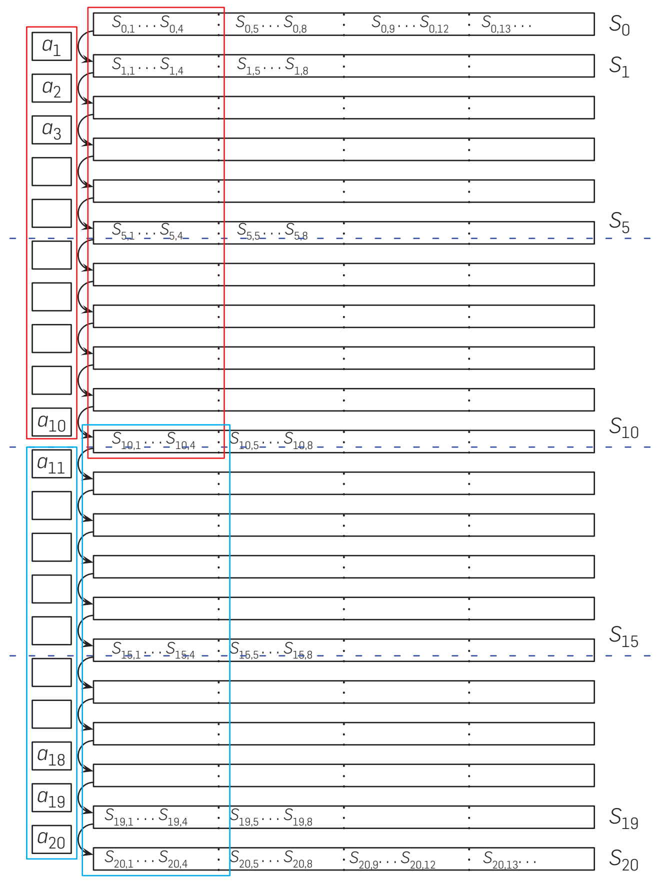


Figure 6. Solving the Rubik's cube using dissection into 4.

Algorithm Dissect4-Rubik

```
Input: An initial state $S_0$ and a final state $S_{20}$
for all $S_{10,1}, S_{10,2}, S_{10,3}, S_{10,4}$ do
    Obtain all candidate $(a_1, a_2, \dots, a_{10})$ satisfying $S_{10,1}, S_{10,2}, S_{10,3}, S_{10,4} = a_{10}(\dots(a_2(a_1(S_{0,1}, S_{0,2}, S_{0,3}, S_{0,4})))\dots)$ by calling PartialMITM$(S_{0,1}, S_{0,2}, S_{0,3}, S_{0,4}, S_{10,1}, S_{10,2}, S_{10,3}, S_{10,4})$
    for all obtained $a_1, a_2, \dots a_{10}$ do
        Compute $S_{10,5}, S_{10,6}, S_{10,7}, S_{10,8} = a_{10}(\dots(a_2(a_1(S_{0,5}, S_{0,6}, S_{0,7}, S_{0,8})))\dots)$
        Store $(S_{10,5}, S_{10,6}, S_{10,7}, S_{10,8}, a_1, a_2, \dots, a_{10})$ in $L_{10}$
    Sort $L_{10}$ according to the values of $S_{10,5}, S_{10,6}, S_{10,7}, S_{10,8}$ in each entry (under some lexicographical order)
    Obtain all candidates $a_{11}, a_{12}, \dots, a_{20}$ satisfying $S_{20,1}, S_{20,2}, S_{20,3}, S_{20,4} = a_{20}(\dots(a_{12}(a_{11}(S_{10,1}, S_{10,2}, S_{10,3}, S_{10,4})))\dots)$ by calling PartialMITM$(S_{10,1}, S_{10,2}, S_{10,3}, S_{10,4}, S_{20,1}, S_{20,2}, S_{20,3}, S_{20,4})$
    for all obtained $a_{11}, a_{12}, \dots a_{20}$ do
        Compute $S_{10,5}, S_{10,6}, S_{10,7}, S_{10,8} = a_{11}^{-1}(\dots(a_{19}^{-1}(a_{20}^{-1}(S_{20,5}, S_{20,6}, S_{20,7}, S_{20,8})))\dots)$
        Search for $S_{10,5}, S_{10,6}, S_{10,7}, S_{10,8}$ in $L_{10}$
        if $S_{10,5}, \dots, S_{10,8}$ are found then
            Obtain the associated $a_1, a_2, \dots, a_{10}$ from $L_{10}$
            if $S_{20} = a_{20}(\dots(a_2(a_1(S_0)))\dots)$ then
                return $a_1, a_2, \dots, a_{20}$ as the solution
```

Procedure PartialMITM

```
Input: An partial initial state $S_{0,1}, S_{0,2}, S_{0,3}, S_{0,4}$ and a partial final state $S_{10,1}, S_{10,2}, S_{10,3}, S_{10,4}$
for all $a_1, a_2, \dots, a_5$ do
    Compute $S_5 = a_5(\dots(a_2(a_1(S_{0,1}, S_{0,2}, S_{0,3}, S_{0,4})))\dots)$
    Store $(S_{5,1}, S_{5,2}, S_{5,3}, S_{5,4}, a_1, a_2, \dots, a_5)$ in a list $L_5$
Sort $L_5$ according to the values of $S_{5,1}, S_{5,2}, S_{5,3}, S_{5,4}$ in each entry (under some lexicographical order)
for all $a_6, a_7, \dots, a_{10}$ do
    Compute $S_{5,1}, S_{5,2}, S_{5,3}, S_{5,4} = a_6^{-1}(\dots(a_9^{-1}(a_{10}^{-1}(S_{10,1}, S_{10,2}, S_{10,3}, S_{10,4})))\dots)$
    Search for $S_{5,1}, S_{5,2}, S_{5,3}, S_{5,4}$ in $L_5$
    if $S_{5,1}, \dots, S_{5,4}$ are found then
        Obtain the associated $a_1, a_2, \dots, a_5$ from $L_5$
        return $a_1, a_2, \dots, a_{10}$ as a candidate solution
```

for S_{10} in the sorted list, and for each match, we output a full solution $a_1, a_2, \dots, a_{20}$.

In order to analyze the algorithm, we fix a value of $S_{10,1}$, $S_{10,2}, S_{10,3}, S_{10,4}$ and estimate the average number of solutions that we expect for the upper (smaller) execution matrix. Namely, we calculate the expected number of action vectors $a_1, \dots, a_{10}$ which transform $S_{0,1}, S_{0,2}, S_{0,3}, S_{0,4}$ into $S_{10,1}$, $S_{10,2}, S_{10,3}, S_{10,4}$. First, we notice that the number of possible action vectors $a_1, \dots, a_{10}$ is about 2^{39}. Each such action vector transforms $S_{0,1}, S_{0,2}, S_{0,3}, S_{0,4}$ into an arbitrary partial state which matches $S_{10,1}, S_{10,2}, S_{10,3}, S_{10,4}$ with probability of about $1/(24^4) \approx 2^{-18.5}$ (which is inverse-proportional to the number of possible values of $S_{10,1}, S_{10,2}, S_{10,3}, S_{10,4}$). Thus, the expected number of solution (that we store in our sorted list) is $2^{39} \times 2^{-18.5} = 2^{20.5}$.

In general, the time complexity of the MITM algorithm is about square root of the search space, and thus its time complexity on the upper execution matrix is about $2^{39/2} = 2^{19.5}$. However, since in this case we could not split the problem into two parts of exactly equal sizes, we expect $2^{20.5}$ solutions (which we enumerate and store), and thus its time complexity is slightly increased to $2^{20.5}$. This is also the expected time complexity of the MITM algorithm on the bottom part (although here we do not store the solutions, but immediately check each one of them). Since we have an outer loop which we execute $24^4 \approx 2^{18.5}$ times, the expected time complexity of the full algorithm is about $2^{18.5+20.5} = 2^{39}$. The expected memory complexity is $2^{20.5}$, required in order to store the solutions for the MITM on the upper part (note that we reuse this memory for each guess of $S_{0,1}, S_{0,2}, S_{0,3}, S_{0,4}$).

4. IMPROVED DISSECTION TECHNIQUES

A closer look at the algorithm presented in Section 3 reveals that the algorithm treats the top and bottom parts of the execution matrix differently. Indeed, while the suggestions from the top part are stored in a table (L_{10} in the example of Figure 6), the suggestions from the bottom part are checked on-the-fly against the table values. As a result, while the number of suggestions in the top part is bounded from above by the size of the memory available for the algorithm, the number of suggestions from the bottom part can be arbitrarily large and generated on-the-fly in an arbitrary order.

This suggests that an asymmetric division of the execution matrix, in which the bottom part is significantly bigger than the top part, may lead to better performance of the algorithm.

In this section we show that this is indeed the case. As the algorithm for untangling the Rubik's cube presented in Section 3 is already practical on a PC so that there is no significant value in further improving it, we choose another classical search problem, known as the *combinatorial partition problem* to be our running example in this section.

The problem is defined as follows. We are given a set of n integers, $U = \{x_1, x_2, \ldots, x_n\}$. Our goal is to partition U into two complementary subsets U_1, U_2 whose elements sum up to the same number, that is,

$$\sum_{x_i \in U_1} x_i = \sum_{x_j \in U_2} x_j. \qquad (1)$$

The combinatorial partition problem is known to be NP-complete,[4] and hence, one cannot expect a sub-exponential solution in general. Nevertheless, there are various techniques which allow to find a solution efficiently in various cases, especially when there exist many partitions (U_1, U_2) which satisfy Equation (1). We thus consider the "hardest" instance of the problem, in which each of the x_is is of n digits in binary representation (i.e., $x_i \approx 2^n$). In this case, at most a few solutions (U_1, U_2) are expected to exist, and no sub-exponential algorithms for the problem are known. For the sake of simplicity, we focus on the *modular* variant of the problem, in which Equation (1) is slightly altered to

$$\sum_{x_i \in U_1} x_i \equiv \sum_{x_j \in U_2} x_j \pmod{2^n}. \qquad (2)$$

As a specific numeric example, consider the case $n = 112$. In this case, checking all 2^{112} possible partitions is, of course, completely infeasible. The standard MITM algorithm allows to reduce the time complexity to 2^{56}, but it increases the space complexity to 2^{56} which is currently infeasible. As we show below, the problem can be represented as a bicomposite problem, and hence, the technique of Section 3 can be applied to obtain the better tradeoff of $T = 2^{56}$ and $S = 2^{28}$. While these numbers are almost practical, the relatively large amount of required memory disallows the use of FPGAs in the computation, which makes it barely feasible. We show below that by an *asymmetric* variant of the dissection algorithm, we are able to obtain the complexities $T = 2^{64}$ (which is only 2^8 times larger) and $S = 2^{16}$ (which is 2^{12} times smaller), which allow for a significantly faster computation using memory-constrained FPGAs. Note that the asymmetric dissection algorithm outperforms the symmetric one by a factor of 16 according to the complexity measure $S \times T$ (the complexities are 2^{80} vs. 2^{84}). Such a factor, while not extremely big, can make a difference in practical scenarios.

4.1. Representing combinatorial partition as a bicomposite search problem

In order to apply dissection algorithms to the combinatorial partition problem, we have to find a way to represent it as a bicomposite search problem.

First, we represent it as a composite problem. We treat the problem of choosing the partition (U_1, U_2) as a sequence of n atomic decisions, where the ith decision is whether to assign $x_i \in U_1$ or $x_i \in U_2$. We introduce a counter C which is initially set to zero, and then at the ith step, if the choice is $x_i \in U_1$ then C is replaced by $C + x_i \pmod{2^n}$, and if the choice is $x_i \in U_2$, C is replaced by $C - x_i \pmod{2^n}$. Note that the value of C after the nth step is $\sum_{x_i \in U_1} x_i - \sum_{x_j \in U_2} x_j \pmod{2^n}$, and hence, the sequence of choices leads to the desired solution if and only if the final value of C is zero.

In this representation, the partition problem has all the elements of a composite problem: an initial state ($C_{\text{initial}} = 0$), a final state ($C_{\text{final}} = 0$), and a sequence of n steps, such that in each step, we have to choose one of two possible atomic actions. Our goal is to find a sequence of choices which leads from the initial state to the final state. In terms of the execution matrix, we define S_i to be the value of C after the ith step (which is an n-bit binary number) and a_i to be the action transforming S_{i-1} to S_i, whose possible values are either $C \leftarrow C + x_i \pmod{2^n}$ or $C \leftarrow C - x_i \pmod{2^n}$.

The second step is to represent the problem as a bicomposite problem. The main observation we use here is the fact that for any two integers a, b, the mth least significant bit (LSB) of $a + b \pmod{2^n}$ depends only on the m LSBs of a and b (and not on their other digits). Hence, if we know the m LSBs of S_{i-1} and the action a_i, we can compute the m LSBs of S_i.

Using this observation, we define $S_{i,j}$ to be the jth LSB of S_i. This leads to an n-by-n execution matrix $S_{i,j}$ for $i, j \in 1, 2, \ldots, n$ with the property that if we choose any rectangle within the execution matrix *which includes the rightmost column of the matrix*, knowledge of the substates $S^j_{i-1}, S^{j+1}_{i-1}, \ldots, S^k_{i-1}$ along its top edge and knowledge of the actions $a_i, a_{i+1}, \ldots, a_\ell$ to its right suffices in order to compute the substates $S^j_\ell, S^{j+1}_\ell, \ldots, S^k_\ell$ along its bottom edge.

Note that the condition satisfied by our execution matrix is weaker than the condition given in the definition of a bicomposite problem, since in our case, the "rectangle" property holds only for rectangles of a certain kind and not for all rectangles. However, as we show in the next subsection, even this weaker property is sufficient for applying all dissection algorithms we present.[b]

4.2. Dissection algorithm for the combinatorial partition problem

The basic idea in the algorithm is to divide the state matrix into seven (!) parts of $n/7$ steps each, where three parts belong to the top part S^t and four parts belong to the bottom

[b] For the sake of simplicity, we disregard the issue of carries. We note that in order to know all the carries required to execute our dissection algorithms, we guess the values of $S_{i,j}$ from the least significant bit to the most significant bit. For more details, refer to the extended version of this paper.[2]

part S^b. The partition is obtained by enumerating the $2n/7$ LSBs of the state $S_{3n/7}$.

For each value v of these bits, we perform a simple MITM algorithm in the top part, which yields about $2^{3n/7} \times 2^{-2n/7} = 2^{2n/7}$ possible combinations of actions $a_1, a_2, \dots, a_{3n/7}$ which lead to a state $S_{3n/7}$ whose $2n/7$ LSBs equal to the vector v. For each of these combinations, we compute the full value of the state $S_{3n/7}$. The resulting values of $S_{3n/7}$ are stored in a table, along with the corresponding combinations of $a_1, a_2, \dots, a_{3n/7}$.

Then we consider the bottom part, and apply to it the dissection algorithm described in Section 3 (thus, dividing it into four chunks of $n/7$ steps each). This results in $2^{4n/7} \times 2^{-2n/7} = 2^{2n/7}$ possible combinations of actions $a_{(3n/7)+1}, a_{(3n/7)+2}, \dots, a_n$ which lead (in the inverse direction) to a state $S_{3n/7}$ whose $2n/7$ LSBs equal to the vector v. For each of these combinations, we compute the full value $S_{3n/7}$ and compare it to the values in the table. If a match is found, this means that the corresponding sequences $\{a_1, a_2, \dots, a_{3n/7}\}$ and $a_{(3n/7)+1}, a_{(3n/7)+2}, \dots, a_n$ match to yield a solution of the problem. Note that if a solution exists, then our method must find it, since it actually goes all over all possible combinations of actions (though, in a sophisticated way). The pseudocode of the algorithm is given in Figure 7.

The memory complexity of the algorithm is $O(2^{n/7})$, as both the standard MITM algorithm for the top part and the dissection algorithm for the bottom part have this complexity. (For the bottom algorithm, the complexity is $(2^{4n/7})^{1/4} = 2^{n/7}$.)

The time complexity is $2^{4n/7}$. Indeed, the enumeration in the state $S_{3n/7}$ is performed over $2^{2n/7}$ values, both the standard MITM algorithm for the top part and the dissection algorithm for the bottom part require $2^{2n/7}$ time, and the remaining $2^{2n/7}$ possible combinations of $a_{(3n/7)+1}, a_{(3n/7)+2}, \dots, a_n$ are checked instantly. This leads to time complexity of $2^{2n/7} \times 2^{2n/7} = 2^{4n/7}$.

In the special case of $n = 112$, each of the seven chunks consists of 16 steps, the enumeration is performed on the 28 LSBs of the state S_{42}, the memory complexity is $2^{n/7} = 2^{16}$, and the time complexity is $2^{4n/7} = 2^{64}$.

4.3. Advanced dissection algorithms

The algorithms presented in Section 3 and in this section are the two simplest dissection algorithms, which demonstrate the general idea behind the technique. In the extended version of the paper,[2] we present more advanced dissection algorithms, which include division of the matrix to "exotic" numbers of parts, such as 11 and 29, and show the optimality of such choices within our general framework.

So far we only considered search algorithms which are not allowed to fail (i.e., if there are any solutions to the problem then our algorithm will always find all of them, but its running time may be longer than expected if the instances are not randomly chosen or if the number of solutions is too large). In Dinur et al.,[2] we also consider algorithms which may fail to find a solution with a small probability, and show how to improve the efficiency of our algorithms in this case by combining them with a classical technique called *parallel collision search*, devised by Wiener and van Oorschot in 1996.[7]

Figure 7. Solving the partitioning problem using dissection into 7.

Algorithm Dissect7-Partition

```
Input: $U = \{x_1, x_2, \dots x_n\}$
for all $S_{3n',1}, S_{3n',2}, \dots, S_{3n',2n'}$ ($2n'$ LSBs of $S_{3n'}$) do
  call PartialMITM($S_{0,1}, S_{0,2}, \dots, S_{0,2n'}, S_{3n',1}, S_{3n',2}, \dots, S_{3n',2n'}, 3n'$)
  for all obtained $a_1, a_2, \dots, a_{3n'}$ do
    Compute the $5n'$ MSBs of $S_{3n'} = a_{3n'}(\dots(a_2(a_1(S_0)))\dots)$
    Store ($S_{3n'}, a_1, a_2, \dots, a_{3n'}$) in $L_{3n'}$
  Sort $L_{3n'}$ according to the values of $S_{3n'}$
  for all $S_{5n',1}, S_{5n',2}, \dots, S_{5n',n'}$ do
    call PartialMITM($S_{3n',1}, S_{3n',2}, \dots, S_{3n',n'}, S_{5n',1}, S_{5n',2}, \dots, S_{5n',n'}, 2n'$)
    for all obtained $a_{3n'+1}, a_{3n'+2}, \dots, a_{5n'}$ do
      Compute the $n'$ bits $S_{5n',n'+1}, S_{5n',n'+2}, \dots, S_{5n',2n'} = a_{5n'}(\dots(a_{3n'+2}(a_{3n'+1}(S_{3n',n'+1}, S_{3n',n'+2}, \dots, S_{3n',2n'})))\dots)$
      Store ($S_{5n'}, a_{3n'+1}, a_{3n'+2}, \dots, a_{5n'}$) in $L_{5n'}$
    Sort $L_{5n'}$ according to the values of $S_{5n'}$
    call PartialMITM($S_{5n',1}, S_{5n',2}, \dots, S_{5n',n'}, S_{7n',1}, S_{7n',2}, \dots, S_{7n',n'}, 2n'$)
    for all obtained $a_{5n'+1}, a_{5n'+2}, \dots, a_{7n'}$ do
      Compute $S_{5n',n'+1}, S_{5n',n'+2}, \dots, S_{5n',2n'} = a^{-1}_{5n'+1}(\dots(a^{-1}_{7n'-1}(a^{-1}_{7n'}(S_n)))\dots)$
      Search for $S_{5n'}$ in $L_{5n'}$
      if $S_{5n'}$ value is found then
        obtain $a_{3n'+1}, a_{3n'+2}, \dots, a_{5n'}$ from $L_{5n'}$
        for all obtained $a_{3n'+1}, a_{3n'+2}, \dots, a_{5n'}$ do
          Compute $S_{3n',2n'+1}, S_{3n',2n'+2}, \dots, S_{3n',7n'} = a^{-1}_{3n'+1}(\dots(a^{-1}_{7n'-1}(a^{-1}_{7n'}(S_{7n'})))\dots)$
          Search for $S_{3n'}$ in $L_{3n'}$
          if $S_{3n'}$ value is found then
            obtain $a_1, a_2, \dots, a_{3n'}$ from $L_{3n'}$
            return $a_1, a_2, \dots, a_{7n'}$ as a solution
```

Procedure PartialMITM

```
Input: A partial state $S_{0,1}, S_{0,2}, \dots, S_{0,tn'}$, a partial state $S_{(t+1)n',1}, S_{(t+1)n',2}, \dots, S_{(t+1)n',tn'}$, and "distance" $(t+1)n'$
for all $a_1, a_2, \dots, a_{n'}$ do
  Compute $S_{n',1}, S_{n',2}, \dots, S_{n',tn'} = a_{n'}(\dots(a_2(a_1(S_{0,1}, S_{0,2}, \dots, S_{0,tn'})))\dots)$
  Store ($S_{n',1}, S_{n',2}, \dots, S_{n',tn'}, a_1, a_2, \dots, a_5$) in a list $L_{n'}$
Sort $L_{n'}$ according to the values of $S_{n',1}, S_{n',2} \dots S_{n',tn'}$
for all $a_{n'+1}, a_{n'+2}, \dots, a_{(t+1)n'}$ do
  Compute $S_{n',1}, S_{n',2}, \dots, S_{n',tn'} = a^{-1}_{n'+1}(\dots(a^{-1}_{(t+1)n'-1}(a^{-1}_{(t+1)n'}(S_{(t+1)n',1}, S_{(t+1)n',2}, \dots, S_{(t+1)n',tn'})))\dots)$
  Search for $S_{n',1}, S_{n',2}, \dots, S_{n',tn'}$ in $L_{n'}$
  if $S_{n',1}, S_{n',2}, \dots, S_{n',tn'}$ are found then
    Obtain the associated $a_1, a_2, \dots, a_{n'}$ from $L_{n'}$
    return $a_1, a_2, \dots, a_{(t+1)n'}$ as a candidate solution
```

5. CONCLUSION

In this paper we introduced the notion of bicomposite search problems, and developed new types of algorithmic techniques called dissection algorithms in order to solve them with improved time and space complexities. We demonstrated how to use these techniques by applying them to two standard types of problems (Rubik's cube and combinatorial partitions). However, some of the most exciting applications of these techniques are in cryptanalysis, which is beyond the scope of this paper. For example, many banks are still using a legacy cryptographic technique called *triple-DES*, which encrypts sensitive financial data by encrypting it three times with three independent keys. A natural question is whether using *quadruple-DES* (which encrypts the data four times with four independent keys) would offer significantly more security due to its longer key and more complicated encryption process. By using our new dissection techniques, we can show the surprising result that finding the full key of quadruple-DES could be achieved with essentially the same time and space complexities as finding the full key of the simpler triple-DES encryption scheme, and thus there is no significant security advantage in upgrading triple-DES to quadruple-DES.

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